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Torque and energy considerations for a magnet in a magnetic liquid

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Abstract. The calculation of the torque T on a permanent ellipsoidal magnet immersed in a magnetic liquid is more delicate than one might be inclined to expect. This problem, for instance, has a bearing on fundamental discussions carried out in the past about the choice between different formulations of electromagnetic theory. The present paper analyses the calculation of T anew, emphasizing the relation to, and the explanation of, the important experiment carried out by Whitworth and Stopes-Roe in 1971. The outcome of the experiment is explained in terms of standard magnetostatics, based upon use of the scalar potential ϕ. The relation to the so-called Kennelly–Sommerfeld controversy is commented upon. The physical interpretation of the situation is elucidated by carrying out in detail an examination of the total energy balance in the special case of spherical magnet geometry.

1. Introduction

The purpose of the present paper is to discuss some fundamental questions related to the calculation of the torque on a solid ellipsoidal permanent magnet immersed in a magnetic liquid when it is acted upon by a transverse external magnetic field. The problem is more delicate than one might be inclined to think beforehand. It was discussed in a lively manner in the literature some years ago; we shall not enter into a historical discussión here but shall be primarily interested in the outcome of the important experiment of Whitworth and Stopes-Roe (WS) [1]. In this experiment the torque on a long thin permanent magnet immersed in different magnetic liquids was measured. The conclusion made by WS was at first sight counter-intuitive: the torque was claimed to depend on the far-distance magnetic field H_0 in the liquid, rather than on the magnetic induction $B_0 = \mu_0\mu_e H_0$, μ_e being the relative permeability of the liquid. (One reason why the WS conclusion may appear strange is that it is the magnetic induction which is generally known to be the basic field quantity since it is the average of the microscopic field over physically infinitesimal volumes.) The interpretation of the WS experiment was reconsidered by Lowes [2] and Page [3]; in particular the extensive paper by Lowes is important in our context. However, by making a data search we have not been able to find more recent papers referring to the WS experiment, except for the review paper of Byrne [4].

In the following we shall show how the outcome of the WS experiment is explainable using the standard theory of magnetostatics, and we shall focus attention on the role played by the fictitious magnetic surface charge densities. Also, we shall show how it is physically instructive to take the energy balance of the magnetic field into consideration.

Let us discuss the situation quantitatively, referring to figure 1. The semiaxes of the ellipsoidal magnet are a, b, c, where it is assumed that $a \geqslant b \geqslant c$. When the surrounding

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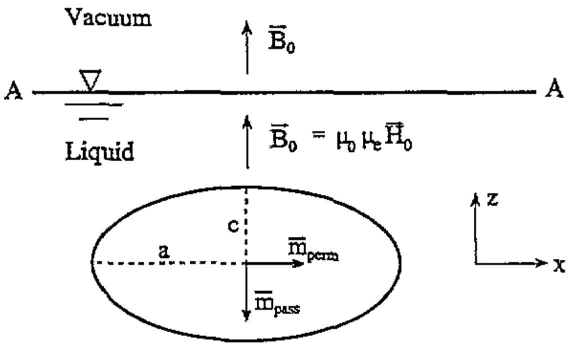

Figure 1. Sketch of the horizontally placed magnet in the transverse field. The direction of m_{pass} presupposes that $\mu_e > 1$.

liquid is present, the magnet possesses a permanent magnetic moment $\boldsymbol{m}_{perm} = m_{perm}\boldsymbol{e}_x$, directed along the horizontal x axis. The value of m_{perm} (dependent on μ_e) follows from the boundary conditions at the magnet surface (see equation (27) below). What is known initially is the uniform permanent magnetization $\boldsymbol{M}_0 = M_0\boldsymbol{e}_x$ when the magnet is surrounded by a vacuum; in that case, the magnetic moment is

$$\boldsymbol{m}_0 = \int \boldsymbol{M}_0 \, \mathrm{d}V = \boldsymbol{M}_0 V \tag{1}$$

where $V = \frac{4}{3}\pi abc$ is the magnet volume. The material in the magnet is taken to be homogeneous and isotropic, and it is moreover taken to possess no induced magnetization properties. That is, the relative 'reversible' permeability (in contrast with [2]) is put equal to unity.

Assume now that there is a transverse magnetic field present, directed along the z axis. At large distances from the magnet the induction is constant, equal to $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$. We shall assume that $\boldsymbol{B}_0$ is produced by external current coils in vacuum (air) above the free surface (line A–A in figure 1), in accordance with the set-up in the WS experiment. The free surface is situated so far from the magnet that no boundary effects from the surface need to be taken into account. We assume that the magnet is kept at rest all the time, and that different magnetic liquids are successively filled into the container. When the current in the coils is always the same, it follows from the continuity of the normal induction across the surface A–A that the far-distant induction B_0 within the liquid is also the same, irrespective of the value of μ_e in the liquid.

Let $\boldsymbol{T}$ denote the torque on the magnet. From simple theory pertaining to the case of linear soft (non-permanent) magnetic media we know that the torque density equals $\boldsymbol{M} \times \boldsymbol{B}$, where $\boldsymbol{M}$ and $\boldsymbol{B}$ are local quantities. In the present case it is evident that $\boldsymbol{T}$ must be equal to $\boldsymbol{m}_0 \times \boldsymbol{B}_0$ times some μ_e-dependent factor. The core of the problem is to determine the factor. The merit of the WS experiment was to show that, in the case of a needle-shaped magnet, the torque was proportional to the field $\boldsymbol{H}_0$:

$$\boldsymbol{T} \propto \boldsymbol{m}_0 \times \boldsymbol{H}_0 = \frac{1}{\mu_0\mu_e}\boldsymbol{m}_0 \times \boldsymbol{B}_0. \tag{2}$$

This meant that, upon interchange of magnetic liquid, T varied inversely proportional to μ_e.

Both [1] and [2] have a bearing on the so-called Kennelly and Sommerfeld formulations of electromagnetism. Let us therefore briefly review this point. We start from the relation

$$B = \mu_0 H + J = \mu_0(H + M) \tag{3}$$

which implies that $J = \mu_0 M$. Here M is the total magnetization, defined as the sum of the permanent and the passive contributions. The quantities J and M are associated with the Kennelly and the Sommerfeld formulations, respectively. It becomes most natural to define the respective total moments of the magnet as volume integrals:

$$j = \int J \, \mathrm{d}V \qquad m = \int M \, \mathrm{d}V \tag{4}$$

implying that

$$j = \mu_0 m. \tag{5}$$

Now consider a long thin magnet immersed in a magnetic liquid: according to Kennelly the torque is predicted to be

$$T = j_0 \times H_0 \qquad \text{(Kennelly)} \tag{6}$$

where j_0 is defined as $j_0 = \int J_0 \, \mathrm{d}V = \mu_0 m_0$, m_0 meaning the same permanent magnetic moment as in equation (1). The torque prediction according to Sommerfeld is

$$T = m_0 \times B_0 \qquad \text{(Sommerfeld).} \tag{7}$$

The physical predictions of (6) and (7) are thus different. Equation (6) is seen to be in agreement with the outcome of the WS experiment (see equation (2)). For this reason, WS claimed their experiment to support the Kennelly formulation only.

The interpretation given by Lowes was, however, different; he claimed that both formulations agree with the WS experiment. In his analysis, two effective magnetic moments j_e and m_e were introduced, defined by the formula

$$\phi = \frac{j_e \cdot \hat{r}}{4\pi \mu_0 \mu_e r^2} \tag{8}$$

for the scalar potential ϕ, and by the formula

$$A = \frac{\mu_0 \mu_e m_e \times \hat{r}}{4\pi r^2} \tag{9}$$

for the vector potential A, respectively. The reason why he claimed the two formulations to be equivalent was that the torque could be expressed in either of the following two ways:

$$T = j_e \times H_0 = m_e \times B_0. \tag{10}$$

Here the first equation corresponds to the Kennelly formulation; the second equation corresponds to the Sommerfeld formulation.

This point is, however, surprising; equation (10) implies that

$$j_e = \mu_0 \mu_e m_e \tag{11}$$

which is in conflict with the general equation (5) above. One may wonder whether Lowes' formal definitions of $\boldsymbol{j}_e$ and $\boldsymbol{m}_e$ in reality tend to obscure the physical significance of the WS experiment. This was one of the reasons why we found it desirable to analyse the WS experiment anew. We shall in the following go through the calculation of $\boldsymbol{T}$ in some detail and also supply the treatment by an analysis of the energy balance.

As is known, magnetostatic theory can equally well be formulated in terms of the vector potential $\boldsymbol{A}$ produced by volume and surface currents as in terms of the scalar potential ϕ produced by effective volume and surface magnetic charges. We follow the latter approach here. When no volume charges are present, the effective surface charge density σ_M is [5]

$$\sigma_M = \boldsymbol{n}\cdot\boldsymbol{M}^- - \boldsymbol{n}\cdot\boldsymbol{M}^+. \tag{12}$$

Here $\boldsymbol{n}$ is the outward normal, and the superscripts − and + refer to the inner and outer magnet surfaces, respectively. The magnetic moment is the integral

$$\boldsymbol{m} = \int \boldsymbol{r}\sigma_M \,\mathrm{d}S \tag{13}$$

taken over the surface. We note that there are in principle four different contributions to $\boldsymbol{m}$; the moment can arise from effective magnetic charges residing on firstly the inner or secondly the outer surface, associated with thirdly the permanent or fourthly the passive parts of the total magnetization $\boldsymbol{M}$.

2. Permanent magnet: no external field

We refer to figure 1, put the external field $\boldsymbol{B}_0$ equal to zero and recall that the magnetization $\boldsymbol{M}_0$ per definition relaxes to a vacuum region around the magnet. When $a > b > c$ it is convenient to introduce ellipsoidal coordinates ξ, η, ζ, given as the three roots for s of the equation [2, 5, 6]

$$\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s} = 1. \tag{14}$$

The surface of the magnet is at $\xi = 0$. We also define the quantity

$$F_a(\xi) = \tfrac{1}{2}abc \int_\xi^\infty \frac{\mathrm{d}s}{(s+a^2)R_s} \tag{15}$$

where $R_s = [(s+a^2)(s+b^2)(s+c^2)]^{1/2}$, similar definitions holding for the other axes b and c. The demagnetizing factors are $N_i = F_i(0)$ with $i = a, b, c$, satisfying the equation $N_a + N_b + N_c = 1$.

To find the distribution of fields on the inside and on the outside of the magnet surface, one has to take into account the boundary conditions. From Lowes' [2] paper, and further references therein, we have on the inside ($\chi_e = \mu_e - 1$ is the magnetic susceptibility)

$$\boldsymbol{H}^- = -\boldsymbol{M}_0 \frac{N_a}{\mu_e - N_a\chi_e} \tag{16}$$

$$\boldsymbol{B}^- = \mu_0 \boldsymbol{M}_0 \frac{(1-N_a)\mu_e}{\mu_e - N_a\chi_e} \tag{17}$$

$$\boldsymbol{M}^- = \boldsymbol{M}_0 \frac{N_a + (1-N_a)\mu_e}{\mu_e - N_a\chi_e}. \tag{18}$$

On the outside, $\boldsymbol{H}^+ = -\nabla\phi^+$, where

$$\phi^+ = M_0 x \frac{F_a(\xi)}{\mu_e - N_a\chi_e}. \tag{19}$$

The external magnetic induction is $\boldsymbol{B}^+ = \mu_0\mu_e\boldsymbol{H}^+$, and the external magnetization is $\boldsymbol{M}^+ = \chi_e\boldsymbol{H}^+$. It is also useful to note that, on the outer surface,

$$\boldsymbol{H}^+(\xi = 0) = \frac{-N_a\boldsymbol{M}_0 + (\boldsymbol{M}_0\cdot\boldsymbol{n})\boldsymbol{n}}{\mu_e - N_a\chi_e}. \tag{20}$$

This is a result that can be found by using (15) and (19), the partial derivatives of ξ at the magnet surface given by

$$\begin{Bmatrix} \partial\xi/\partial x \\ \partial\xi/\partial y \\ \partial\xi/\partial z \end{Bmatrix}_{\xi=0} = \frac{2}{\eta\zeta}\begin{Bmatrix} xb^2c^2 \\ yc^2a^2 \\ za^2b^2 \end{Bmatrix} \tag{21}$$

which follow from (14) and the general equation

$$\frac{\eta\zeta}{a^2b^2c^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \tag{22}$$

and finally by observing that the components of $\boldsymbol{n}$ at the surface are

$$\begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \frac{1}{\sqrt{\eta\zeta}}\begin{Bmatrix} xbc/a \\ yca/b \\ zab/c \end{Bmatrix}. \tag{23}$$

A remarkable property of the above formalism is that the inside magnetization $\boldsymbol{M}^-$ depends also on the outside permeability μ_e. Let us calculate the effective surface charge densities; on the inside,

$$\sigma_M^- = \boldsymbol{n}\cdot\boldsymbol{M}^- = \frac{N_a + (1 - N_a)\mu_e}{\mu_e - N_a\chi_e}(\boldsymbol{M}_0\cdot\boldsymbol{n}) \tag{24}$$

and on the outside

$$\sigma_M^+ = -\boldsymbol{n}\cdot\boldsymbol{M}^+ = \frac{(N_a - 1)\chi_e}{\mu_e - N_a\chi_e}(\boldsymbol{M}_0\cdot\boldsymbol{n}). \tag{25}$$

The total surface charge density is thus

$$\sigma_M = \sigma_M^- + \sigma_M^+ = \frac{\boldsymbol{M}_0\cdot\boldsymbol{n}}{\mu_e - N_a\chi_e}. \tag{26}$$

The geometric form of the ellipsoid is so far assumed to be arbitrary (recall, however, that our conventions when using ellipsoidal coordinates are that the inequalities $a > b > c$ are fulfilled). Now proceeding to calculate the magnetic moment $\boldsymbol{m}$ using (13), we shall for simplifying reasons assume that $a \geqslant b = c$, so that the ellipsoid degenerates into a prolate spheroid. Because of symmetry, only the longitudinal component of $\boldsymbol{m}$ survives. We can now avoid ellipsoidal coordinates. In terms of Cartesian coordinates we have simply $n_x\,\mathrm{d}S = 2\pi(c/a)^2 x\,\mathrm{d}x$ and we get, upon insertion of (26),

$$\boldsymbol{m}_{perm} = \frac{\boldsymbol{m}_0}{\mu_e - N_a\chi_e} \tag{27}$$

(we supply an extra index *perm* to emphasize that it is the moment of the permanent magnet). Note that this result is a consequence of the surface charges on the inside surface, as well as on the outside surface, of the magnet.

In the special case when the spheroid degenerates into a long thin rod, we have $N_a \to 0$, $N_b = N_c \to \frac{1}{2}$. Then (27) yields

$$\boldsymbol{m}_{perm} = \boldsymbol{m}_0/\mu_e \tag{28}$$

showing the significance of the outside medium even in this simple case.

3. Passive magnet in transverse field

We refer again to figure 1, put $M_0 = \mathbf{0}$ and assume the transverse external field $B_0 = B_0 e_z$ to be present. The solution of the field problem can everywhere be written as $H = -\nabla\phi$, where on the inside

$$\phi^- = -H_0 z \frac{\mu_e}{\mu_e - N_c \chi_e} \tag{29}$$

and on the outside

$$\phi^+ = -H_0 z \left[1 + F_c(\xi) \frac{\chi_e}{\mu_e - N_c \chi_e}\right]. \tag{30}$$

Again, we have made use of results derived in [2]. We write down the expressions for the internal field given by

$$H^- = H_0 \frac{\mu_e}{\mu_e - N_c \chi_e} \tag{31}$$

and for the external field at the magnet surface given by

$$H^+(\xi = 0) = \frac{H_0}{\mu_e - N_c \chi_e}\left[\mu_e e_z - \frac{zab}{c} \frac{\chi_e}{\sqrt{\eta\zeta}} n\right] \tag{32}$$

where in the last derivation we made use of (21) and (23). Equation (32) can be expressed in terms of the x, y, z coordinates using equation (22). Equation (32) is analogous to equation (20) holding for the permanent magnet.

Let us calculate the internal and external magnetic surface charge densities. As $\sigma_M^- = n \cdot M^-$ and $\sigma_M^+ = -n \cdot M^+$ in general, and as $M^- = \mathbf{0}$ (no soft internal magnetization) and $M^+ = \chi_e H^+$ in the present case, we obtain using (32) and (23)

$$\sigma_M^- = 0 \tag{33}$$

$$\sigma_M^+ = -\frac{\chi_e}{\mu_e - N_c \chi_e}(H_0 \cdot n). \tag{34}$$

The total surface charge density for the passive magnet becomes thus

$$\sigma_M = -\frac{\chi_e}{\mu_e - N_c \chi_e}(H_0 \cdot n). \tag{35}$$

Now considering the passive magnetic moment m_{pass} we assume, as in the previous section, that the magnet takes the simple form of a prolate spheroid with $b = c$. Define the distance ρ by $\rho^2 = y^2 + z^2$, and let φ be the polar angle in the y–z plane. The surface element on the magnet surface becomes

$$\mathrm{d}S = \rho\sqrt{1 + (\mathrm{d}\rho/\mathrm{d}x)^2}\,\mathrm{d}\varphi\,\mathrm{d}x = \rho\sqrt{1 + (xc^2/\rho a^2)^2}\,\mathrm{d}\varphi\,\mathrm{d}x \tag{36}$$

and the component n_z of the outward normal n is given by (23) and (22). It follows that $n_z\,\mathrm{d}S = z\,\mathrm{d}\varphi\,\mathrm{d}x$. Insertion of (35) into the basic formula (13) then yields $m_x = m_y = 0$, whereas the vertical component m_z is non-vanishing. Altogether we obtain for the passive magnet

$$m_{pass} = -\frac{\chi_e}{\mu_e - N_c \chi_e} H_0 \int_0^a 2\pi\rho^2\,\mathrm{d}x = -\frac{\chi_e}{\mu_e - N_c \chi_e}(H_0 V) \tag{37}$$

where $V = (4\pi/3)ac^2$ is the volume. The limiting case of a long thin horizontal rod ($N_c \to \frac{1}{2}$) may be taken in the same way as in the previous section (see equation (28)).

4. The torque

Place an arbitrary fictitious surface S outside the surface of the ellipsoidal magnet. The torque on the magnet is given as the surface integral

$$\boldsymbol{T} = \mu_0\mu_e \int \boldsymbol{r} \times \left[\boldsymbol{H}(\boldsymbol{H}\cdot\boldsymbol{n}) - \tfrac{1}{2}H^2\boldsymbol{n}\right] \mathrm{d}S \tag{38}$$

where $\boldsymbol{H}$ is the total field on S and $\boldsymbol{n}$ the outward normal. The reason why the position of S can be taken to be arbitrary is that the magnetic force density in the fluid is equal to zero. It is mathematically simplifying to exploit this fact and to choose S to be a spherical surface, far from the origin, so that $\boldsymbol{n} \to \hat{\boldsymbol{r}}$. At large distances ($\xi \gg a, b, c$) we have $R_\xi \simeq \xi^{3/2}$, and thus

$$F_a(\xi) \simeq F_b(\xi) \simeq F_c(\xi) \simeq \frac{V}{4\pi}\xi^{-3/2} \tag{39}$$

according to (15). Also, $r^2 \simeq \xi$ according to (14).

From the structure of (38) it is clear that one gets a non-vanishing contribution to the torque only when $\boldsymbol{H}_0$ interacts with fields of order r^{-3}. These fields are precisely the dipole fields. There are two fields of this kind: one is due to the permanent magnetic moment $\boldsymbol{m}_{perm}$ in (27); the other is due to the induced magnetic moment $\boldsymbol{m}_{perm}$ in (37). Altogether, the total potential at large distances is

$$\phi^+ = -\boldsymbol{H}_0\cdot\boldsymbol{r} + \frac{\boldsymbol{m}\cdot\hat{\boldsymbol{r}}}{4\pi r^2} \tag{40}$$

and the total field is

$$\boldsymbol{H}^+ = \boldsymbol{H}_0 + \frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}}{4\pi r^3} \tag{41}$$

where $\boldsymbol{m}$ is the total magnetic moment:

$$\boldsymbol{m} = \frac{\boldsymbol{m}_0}{\mu_e - N_a\chi_e} - \frac{\chi_e}{\mu_e - N_c\chi_e}(\boldsymbol{H}_0 V). \tag{42}$$

The last term in (38) does not contribute, and we may write the torque as an integral over solid angles:

$$\boldsymbol{T} = -\frac{\mu_0\mu_e}{4\pi}\int_{sphere} [(\hat{\boldsymbol{r}}\times\boldsymbol{m})\boldsymbol{H}_0\cdot\hat{\boldsymbol{r}} - 2(\hat{\boldsymbol{r}}\times\boldsymbol{H}_0)\boldsymbol{m}\cdot\hat{\boldsymbol{r}}]\,\mathrm{d}\Omega \tag{43}$$

where $\mathrm{d}S = r^2\,\mathrm{d}\Omega$. Here we made use of (41), and retained only the non-vanishing r-independent terms. It is seen that the passive magnetic moment (the last term in (42)) does not contribute to the torque. This behaviour could also have been expected beforehand, because of symmetry. The only remaining contribution is that arising from the permanent moment, and when expressing (43) in terms of usual spherical coordinates r, θ, φ we obtain after a brief calculation

$$\boldsymbol{T} = \boldsymbol{m}_{perm}\times\boldsymbol{B}_0 = \frac{\boldsymbol{m}_0\times\boldsymbol{B}_0}{\mu_e - N_a\chi_e}. \tag{44}$$

This is the main result of the analysis. Comparison with (26) shows that it is the total (inside plus outside) magnetic surface charge density associated with the permanent magnet that is responsible for the torque; the surface charge density (35) associated with the passive magnet does not play any role here.

For a long horizontal magnet we get

$$T = \frac{1}{\mu_e} m_0 \times B_0 = \mu_0 m_0 \times H_0 \tag{45}$$

which agrees with (2) and therefore agrees also with the outcome of the WS experiment. The applicability of (45) thus presupposes this particular geometrical form of the magnet. The simple proportionality between T and H_0 would not hold, for instance, if the magnet instead were a sphere. It would be of some interest to repeat the torque experiment under such circumstances.

5. Energy considerations: spherical geometry

As we have seen, it is the permanent magnet's inside surface charge density σ_M^- plus the outside charge density σ_M^+ which is essential for the torque on the magnet. Both densities are according to (24) and (25) dependent on the value of μ_e. One may wonder what the physical process is which is responsible for the establishment of these charge densities. Imagine for concreteness that the magnet is initially surrounded by a liquid of permeability μ_e, and that the container is thereafter slowly emptied. The final state corresponds to $\mu_e = 1$, implying that $\sigma_M^- = M_0 \cdot n$, $\sigma_M^+ = 0$. The surface charges thus gradually change when the container is emptied. Obviously there is a physical process of some sort going here.

The purpose of the present section is to stress the following fact: the key process taking place during the change of liquid is the mechanical work exerted by the magnetic volume force density $f = -\frac{1}{2}\mu_0 H^2 \nabla \mu_e$ in the inhomogeneous boundary region at the free surface of the liquid. We shall illustrate this point by analysing the total energy balance for the magnetic field. We shall henceforth assume spherical magnet geometry, with $b = c = a$. This assumption simplifies the mathematics; yet the case is non-trivial enough to demonstrate the essential features of the physics involved. We consider the permanent magnet only (i.e. put $B_0 = 0$). Let us summarize the basic formulae when the magnet is completely surrounded by the liquid; from (16)–(18) we get

$$H^- = -\frac{M_0}{2\mu_e + 1} \tag{46}$$

$$B^- = \mu_0 M_0 \frac{2\mu_e}{2\mu_e + 1} \qquad M^- = M_0. \tag{47}$$

On the outside we get, using (19), since $F_a(\xi) = \frac{1}{3}a^3/r^3$ *for spherical geometry,*

$$H^+ = \frac{1}{4\pi r^3}\left[3(m_{perm} \cdot \hat{r})\hat{r} - m_{perm}\right] \tag{48}$$

with

$$m_{perm} = \frac{3m_0}{2\mu_e + 1}. \tag{49}$$

One merit of the spherical geometry case is thus that the external field has a dipole form for all r. From (24) and (25) we get the surface charge densities

$$\sigma_M^- = \boldsymbol{M}_0 \cdot \boldsymbol{n} \qquad \sigma_M^+ = \frac{-2\chi_e}{2\mu_e + 1}(\boldsymbol{M}_0 \cdot \boldsymbol{n}). \tag{50}$$

With reference to figure 2, we now imagine the (assumed infinite) container to be emptied in the following way. Initially, the liquid fills all the space around the magnet. Then the liquid is slowly pressed outwards, everywhere in the radial direction (gravity is ignored). There are thus three different field regions present, namely the magnet region $r < a$, the annular vacuum region $a < r < b$, and the outer fluid region $r > b$. In the boundary region around $r = b$ there is a radial volume force density which can be written as

$$f_r = -\tfrac{1}{2}\mu_0\big(H_r^2 + H_\theta^2\big)\frac{\mathrm{d}\mu_e}{\mathrm{d}r} = \frac{B_r^2}{2\mu_0}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{\mu_e}\right) - \tfrac{1}{2}\mu_0 H_\theta^2 \frac{\mathrm{d}\mu_e}{\mathrm{d}r} \tag{51}$$

where θ is the polar angle relative to the x axis. Upon integration across the boundary layer, from $r = b-$ to $r = b+$, we obtain the radial surface force density

$$F_r = -\tfrac{1}{2}\mu_0\chi_e\big[\mu_e H_r^2(b+) + H_\theta^2(b+)\big] \tag{52}$$

(this expression could alternatively be derived by starting instead from Maxwell's stress tensor). In order to evaluate F_r, and as the next step also the work exerted during the removal of the liquid, we thus first have to solve the field problem in the three regions mentioned above.

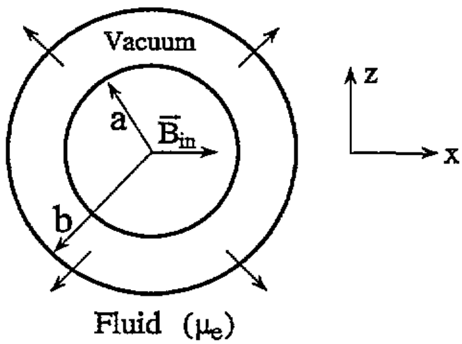

Figure 2. Spherical permanent magnet with radius $r = a$, surrounded by a vacuum region $a < r < b$ and a magnetic liquid region $r > b$.

Within the magnet we have, for an arbitrary value of b, $\boldsymbol{B}_{in} = B_{in}\boldsymbol{e}_x$, where initially $\boldsymbol{B}_{in} = \boldsymbol{B}^-$ as given by (47), and where finally $\boldsymbol{B}_{in} = \frac{2}{3}\mu_0 \boldsymbol{M}_0$ (when all liquid is removed). The corresponding magnetic field is

$$\boldsymbol{H}_{in} = \frac{1}{\mu_0}\boldsymbol{B}_{in} - \boldsymbol{M}_0. \tag{53}$$

Outside the magnet it is most convenient to work in terms of the potentials; in the annular region,

$$\phi = \left(\alpha r + \frac{\beta}{r^2}\right) \cos\theta \tag{54}$$

and, in the outer region,

$$\phi = \frac{\gamma}{r^2} \cos\theta. \tag{55}$$

Here the four quantities B_{in}, α, β and γ are to be determined from the boundary conditions at $r = a$ and $r = b$. A brief calculation yields

$$B_{in} = \tfrac{2}{3}\mu_0 M_0\left(1 + \frac{a^3}{b^3}\frac{\chi_e}{2\mu_e + 1}\right) \tag{56}$$

$$\alpha = -\tfrac{2}{3}M_0\frac{a^3}{b^3}\frac{\chi_e}{2\mu_e + 1} \tag{57}$$

$$\beta = \tfrac{1}{3}M_0 a^3 \qquad \gamma = \frac{M_0 a^3}{2\mu_e + 1}. \tag{58}$$

In fact the surface force density, when expressed as in (52), requires knowledge about the coefficient γ only. Insertion for $H_r(b+)$ and $H_\theta(b+)$ yields

$$F_r = -\tfrac{1}{2}\mu_0\chi_e\frac{\gamma^2}{b^6}\left(4\mu_e \cos^2\theta + \sin^2\theta\right). \tag{59}$$

The work W exerted by this force during the removal of the liquid can now be found:

$$W = -\mu_0\chi_e\pi\gamma^2\int_a^\infty \frac{\mathrm{d}b}{b^4}\int_0^\pi \left(4\mu_e\cos^2\theta + \sin^2\theta\right)\sin\theta\,\mathrm{d}\theta = -\frac{4\pi}{9}\mu_0 M_0^2 a^3\frac{\chi_e}{2\mu_e + 1}. \tag{60}$$

We shall assume that $\mu_e > 1$. The work is then always negative; this is so because the force density $f = -\frac{1}{2}\mu_0 H^2\nabla\mu_e$ in the boundary layer around $r = b$ is directed inwards.

We shall now compare W with the change ΔU in magnetic energy, written as the sum of interior and exterior parts:

$$\Delta U = \Delta U_{int} + \Delta U_{ext}. \tag{61}$$

Consider first the interior part: B_{in} and H_{in} are given by (56) and (53). When the radius of the liquid increases from b to $b + \mathrm{d}b$, the interior magnetic energy density changes by

$$H_{in}\,\mathrm{d}B_{in} = \tfrac{2}{3}\mu_0 M_0^2 a^3\frac{\chi_e}{2\mu_e + 1}\left(\frac{1}{b^4} - \frac{2\chi_e}{2\mu_e + 1}\frac{a^3}{b^7}\right)\mathrm{d}b. \tag{62}$$

Therefore we have

$$\Delta U_{int} = V\int_{b=a}^\infty H_{in}\,\mathrm{d}B_{in} = \frac{8\pi}{27}\mu_0 M_0^2 a^3\frac{\chi_e(\mu_e + 2)}{(2\mu_e + 1)^2}. \tag{63}$$

As regards the exterior shift in energy, we must integrate the magnetic energy density over the exterior region when the liquid is absent, and subtract the corresponding expression when the liquid is present. Using (48) and (49) we obtain

$$\Delta U_{ext} = \frac{\mu_0 m_0^2}{16\pi}\left[1 - \frac{9\mu_e}{(2\mu_e+1)^2}\right]\int_a^\infty \frac{\mathrm{d}r}{r^4}\int_0^\pi \left(3\cos^2\theta + 1\right)\sin\theta\,\mathrm{d}\theta$$
$$= \frac{4\pi}{27}\mu_0 M_0^2 a^3\left[1 - \frac{9\mu_e}{(2\mu_e+1)^2}\right]. \tag{64}$$

It follows from (60)–(64) that

$$\Delta U = -W. \tag{65}$$

The negative work W due to the magnetic surface force when the liquid is pushed out from $r = a$ to $r = \infty$ thus implies an increase in magnetic field energy which is exactly equal to ΔU. This is the physical process that takes place when the container in practice is emptied of one liquid and afterwards filled up by another liquid. In turn, this process has an influence also upon the magnetic surface charges. As equation (50) shows, σ_M^- is in the case of spherical geometry uninfluenced by the external liquid, whereas σ_M^+ is influenced. If the liquid is non-magnetic, σ_M^+ vanishes.

6. Conclusion and final remarks

Let us summarize as follows.

(1) The torque $\boldsymbol{T}$ on an immersed spheroidal magnet is in general given by (44). The permeability μ_e of the liquid is thus of importance here. Note that $\boldsymbol{m}_0$ is defined as in equation (1); it would be the magnetic moment if the magnet were placed in a vacuum. If the magnet is a long thin rod, the formula for $\boldsymbol{T}$ is as in (45), showing the agreement with the outcome of the WS experiment (see equation (2)).

(2) As regards the Kennelly–Sommerfeld controversy, we see that neither the Kennelly prediction (6) nor the Sommerfeld prediction (7) is in general correct, although accidentally the Kennelly prediction is right if the magnet is a thin long rod; cf. (6) and (45). This coincidence is the reason why WS [1] favoured the Kennelly prediction. In our opinion the Kennelly moment density $\boldsymbol{J}$ and moment $\boldsymbol{j}$ (see equations (3) and (4)) ought to be abandoned altogether since these symbols are unnecessary and tend to confuse the physical interpretation.

(3) Our derivation of the torque in (44) is in agreement with the work of Lowes [2], his symbol $\boldsymbol{m}_e$ meaning the same as our $\boldsymbol{m}_{perm}$. He actually argues that the Kennelly and the Sommerfeld approaches are equivalent, assuming that the proper interpretation is given. However, his argument, as shown in section 1, implies the relationship (11) which in our opinion is unfortunate since it conflicts with the natural relationship (5). Again, we wish to emphasize our main attitude, which is to avoid introducing unnecessary quantities in the theory.

We ought perhaps to point out also the following difficulty encountered with Lowes' paper. In our treatment above, we put the relative 'reversible' permeability of the magnet equal to unity. Lowes develops the theory in a more generalized form, allowing for an arbitrary internal permeability, called μ_i. Now equation (19) of [2] implies, in our notation and with our orientation of the magnet axes, that

$$m_{perm} = \frac{m_0}{\mu_e + N_a(\mu_i - \mu_e)}. \tag{66}$$

According to this equation, if the permanent magnet is situated in a vacuum, one should get $m_{perm} = m_0/[1 + N_a(\mu_i - 1)]$. This is a result which is in conflict with the general property $m_{perm} = m_0$ which has to hold for a magnet in vacuum (see equation (1)). The generalization given by Lowes therefore appears questionable at this point. If $\mu_i = 1$, the problem disappears.

(4) We found it convenient to work in terms of the scalar potential ϕ produced by effective surface magnetic charges σ_M; see equation (13) for the magnetic moment. The physical process responsible for the establishment of the μ_e-dependent value of σ_M on the surface of the permanent magnet (equation (26)) is the mechanical work exerted by the magnetic surface forces during emptying and filling of liquids in the container. This point was illustrated by a detailed calculation in section 5.

(5) Because of the applied transverse field H_0 a passive magnetic moment m_{pass} is established, as shown in (37). This moment has no influence upon the torque on the magnet. It should be noted that when $\mu_e > 1$ the direction of m_{pass} is opposite to that of H_0; (see also figure 1). We are here encountering a magnetic analogue of Archimedes' principle.

In fact the situation where non-magnetic monosized spherical particles in the micrometre range are dispersed in a ferrofluid is of considerable interest in modern fundamental studies of collective phenomena. One obtains a system of interacting magnetic dipoles called 'magnetic holes'. The initial experimental discovery was made by Skjeltorp [7] (see also Skjeltorp and co-workers [8, 9] and the thesis of Helgesen [10]). As we have seen above, there cannot be any torque on a single magnetic hole in the static case. However, a bound pair of magnetic holes subjected to a rotating magnetic field will experience a magnetic torque. We shall not here go into further consideration of this point but mention finally that the attractive van der Waals force F_{vdW} between two such holes of radius a, when the minimum distance d between the two spherical surfaces is much less than a, can be written as [11]

$$F_{vdW} = -\frac{\hbar a}{32\pi d^2}\int_0^\infty \mathrm{d}\xi \int_0^\infty \frac{x^2\,\mathrm{d}x}{\Delta^{-2}(\mathrm{i}\xi)\exp x - 1} \tag{67}$$

where Δ is the quantity

$$\Delta(\mathrm{i}\xi) = \frac{\mu_e(\mathrm{i}\xi) - 1}{\mu_e(\mathrm{i}\xi) + 1}. \tag{68}$$

In this case the dispersion of the liquid is thus essential.

References

[1] Whitworth R W and Stopes-Roe H V 1971 *Nature* **234** 31
[2] Lowes F J 1974 *Proc. R. Soc.* A **337** 555
[3] Page C H 1976 *Am. J. Phys.* **44** 1104
[4] Byrne J V 1977 *Proc. IEE* **124** 1089
[5] Stratton J A 1941 *Electromagnetic Theory* (New York: McGraw-Hill)
[6] Landau L D and Lifshitz E M 1984 *Electrodynamics of Continuous Media* 2nd edn (Oxford: Pergamon)
[7] Skjeltorp A T 1983 *Phys. Rev. Lett.* **51** 2306
[8] Skjeltorp A T 1984 *J. Appl. Phys.* **55** 2587
[9] Helgesen G, Pieranski P and Skjeltorp A T 1990 *Phys. Rev. Lett.* **64** 1425
[10] Helgesen G 1990 *Scient. Thesis* University of Oslo
[11] Brevik I and Skurdal H 1993 *Physica* A **199** 412